

1. An important principle of classical mechanics is *Galilean invariance*, which says that physical equations should be invariant with respect to transformation to any reference frame moving at constant speed. We want to examine some important equations of incompressible fluid mechanics in that regard.
- a. Let us first consider the incompressible Navier-Stokes equations. Are these equations Galilei-invariant? Provide a proof for your answer.
 - b. Now use the above equations to *carefully* derive the Bernoulli equation, giving the assumptions that are required for each step. Show under what conditions this equation will be valid everywhere in a flow field.
 - c. Let us now simplify things by considering the case of a steady flow field.
 - * How would you define the concept of a "steady flow field"?
 - * What relevance, if any, does the notion of Galilean invariance have for the above concept?
 - * Discuss Galilean invariance of the steady Bernoulli equation. Why, exactly, does GI break down for this equation, and what does this mean for the physical validity of such an equation?
 - * Can you propose a version of the Bernoulli equation that does not have the above flaw?

2.

- a. Two cylinders of length L enclose a viscous incompressible fluid. If the inner cylinder is at rest and the outer cylinder rotates at a constant angular speed, calculate the torque required to rotate the outer cylinder and that required to hold the inner cylinder at rest.
- b. If the gap between the two cylinders is very small compared to their radius and the inner cylinder is positioned so that its axis is a small distance from the axis of the outer cylinder, develop the expression for the maximum load that can be carried radially by the inner cylinder when a high viscosity fluid fills the gap.
- c. Using the solution for the flow between two concentric rotating cylinders in part a., deduce the velocity distribution created by a circular cylinder which is rotating in an infinite fluid which is otherwise at rest. Compare this result with that for a line vortex of strength Γ ,

$$\Gamma = 2\pi \frac{R_i^2}{\omega_i},$$

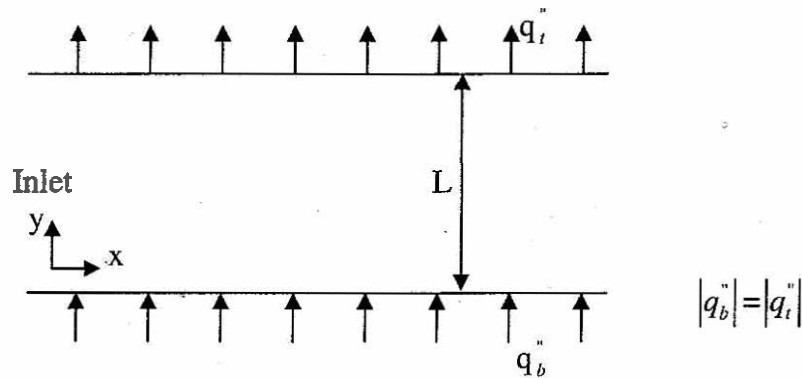
in an inviscid fluid which is at rest at infinity.

HEAT TRANSFER I

Consider fully developed constant-property laminar flow between parallel planes (separated by distance L where L is on the order of a centimeter) with a fully developed temperature profile where heat is transferred to the fluid on one side and out of the fluid on the other at the same rate.

- a) Derive the value of Nusselt number on top and bottom side of the passage.
- b) Sketch the temperature profile at any cross section of the passage.
- c) Suppose the fluid is an oil for which the viscosity varies greatly with temperature, but all the other properties are relatively unaffected by temperature. Is the velocity profile affected? Is the temperature profile affected? Is the Nusselt number affected? Explain.

Note: State all your assumptions explicitly.



Steady state, 2-D momentum, continuity, and thermal energy equations:

$$\rho u \left(\frac{\partial u}{\partial x} \right) + \rho v \left(\frac{\partial u}{\partial y} \right) = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \rho g_x$$

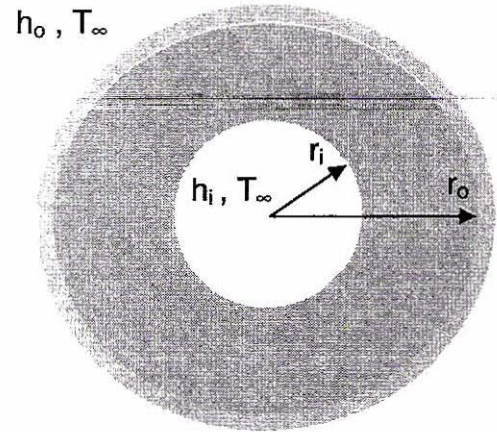
$$\rho u \left(\frac{\partial v}{\partial x} \right) + \rho v \left(\frac{\partial v}{\partial y} \right) = -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \rho g_y$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \left(u \frac{\partial P}{\partial x} + v \frac{\partial P}{\partial y} \right) + \mu \Phi + \dot{q}_{gen}$$

HEAT TRANSFER II

Consider a rotating hollow axle of inner radius r_i and outer radius r_o . The axle sits inside a thin, cylindrical sleeve. The sleeve is of negligible thickness, thus providing negligible resistance to conduction. Assume the axle has thermal conductivity k and rotates at a constant angular velocity ω relative to the sleeve. Assume the sleeve applies a uniform pressure P and a dry friction coefficient μ at the interface with the axle. If heat is lost by convection from both the inner surface of the axle (h_i, T_∞) and the outer surface of the sleeve (h_o, T_∞), what value of r_i results in a minimum temperature at the interface $T(r_o)$?



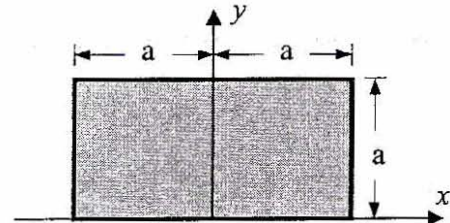
PhD Qualifying Exam Questions Solid Mechanics

Problem 1. Under a certain loading, the following is proposed as the stress distribution in the plate shown.

$$\sigma_x = c_1 x^3 y - 2c_2 xy + c_3 y$$

$$\sigma_y = c_1 xy^3 - 2c_1 x^3 y$$

$$\tau_{xy} = -(3/2)c_1 x^2 y^2 + c_2 y^2 + (1/2)c_1 x^4 + c_4$$



- (a) Is it a permissible stress state? Why?
- (b) Is it a permissible stress state, if the plate is made of a linear elastic material? In light of question (a) is this a redundant question? Explain!
- (c) If your response to (a) and/or (b) is yes, and for the special case when $c_1 = 0$, $c_3 = 0$, $c_4 = -c_2 a^2$, determine and sketch the surface forces that act along the edges $y=0$, $y=a$, and $x=\pm a$.
- (d) Show whether or not the sum of these surface forces (in part c) on the plate are equal to zero, in both the x and y directions (Assume that body forces are negligible).

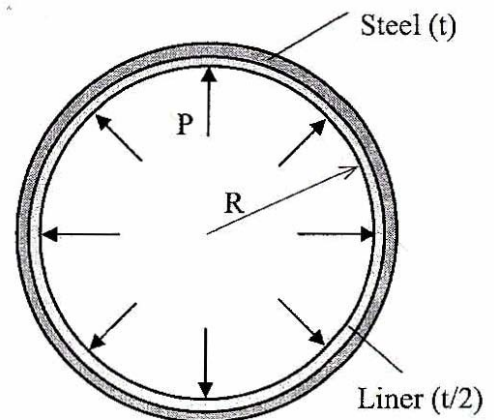
Problem 2. A thin-walled pressure vessel of radius R consists of an outer steel shell of wall thickness t and an inner liner of a non-corrosive metal of wall thickness $t/2$. The elastic moduli and yield stresses of the materials are:

$$\text{Steel: } E_s = 30 \times 10^3 \text{ ksi, } \sigma_{ys} = 80 \text{ ksi}$$

$$\text{Liner: } E_l = 15 \times 10^3 \text{ ksi, } \sigma_{yl} = 20 \text{ ksi}$$

Given that $R/t = 100$ and that the vessel has zero axial stress calculate the following:

- (a) A relationship between the internal pressure P and the vessel radial displacement w .
- (b) The pressure P_y at which the inner layer yields.
- (c) The relationship between P and w for $P > P_y$ if the liner material is elastic-perfectly plastic. Sketch P versus w/R for $P < P_y$ and $P \geq P_y$.



ENGINEERING ANALYSIS I

Consider the temperature distribution due to heat conduction in a circular disk of radius one. The upper semicircular boundary of the disk is maintained at temperature $T = 1$, and the lower semicircular boundary at temperature $T = -1$. In order to obtain the temperature distribution in the disk, consider the following steps.

a) Show that the fractional transformation

$$w = f(z) = \frac{1+z}{1-z}$$

maps the inside of the unit circle in the z -plane to the *right-half plane* of the transformed w -plane. Clearly indicate the boundary conditions on the domain in the w -plane.

b) Solve the Laplace equation for the temperature $\hat{T}(\hat{r}, \hat{\theta})$ in the transformed w -plane. The Laplace equation in polar coordinates is

$$\frac{\partial^2 \hat{T}}{\partial \hat{r}^2} + \frac{1}{\hat{r}} \frac{\partial \hat{T}}{\partial \hat{r}} + \frac{1}{\hat{r}^2} \frac{\partial^2 \hat{T}}{\partial \hat{\theta}^2} = 0,$$

where \hat{r} and $\hat{\theta}$ are the polar coordinates in the w -plane. Note that in polar coordinates, the solution in the right-half plane with the given boundary conditions is independent of \hat{r} , that is $\hat{T} = \hat{T}(\hat{\theta})$.

c) Using the solution in the w -plane, determine the solution $T = T(r, \theta)$ or $T = T(x, y)$ in the z -plane.

d) Consider an alternative solution to that in (a)–(c) for the temperature distribution in a circular disk. Recall that Poisson's integral formula for the solution of the *Dirichlet problem* in a circular disk of radius R is

$$T(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} T(R, \phi) \frac{R^2 - r^2}{R^2 + r^2 - 2Rr \cos(\phi - \theta)} d\phi, \quad r < R,$$

where $T(R, \phi)$ is the temperature distribution on the boundary $|z| = R$. It can be shown that Poisson's integral formula may be written in the form of a series as follows

$$T(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n [a_n \cos(n\theta) + b_n \sin(n\theta)],$$

where the constants a_n and b_n are

$$\begin{aligned}a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} T(R, \phi) d\phi, \\a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} T(R, \phi) \cos(n\phi) d\phi, \\b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} T(R, \phi) \sin(n\phi) d\phi.\end{aligned}$$

Note that when $r = R$, the above series is a Fourier series representation of the temperature on the boundary.

Determine the series solution for $T(r, \theta)$ in the circular disk having the stated boundary conditions.

e) Compare your solutions from (c) and (d) along the real axis.

ENGINEERING ANALYSIS II

a) Using the substitution

$$\frac{dv}{dx} = u$$

in the integral equation

$$\int_0^x (x + \xi)u(\xi) d\xi = x^3$$

and integrating by parts obtain a Volterra equation of the second kind. Find an exact solution beginning with iteration.

b) By differentiating the given integral equation, convert it to a differential equation. Verify that the solution obtained in (a) satisfies the differential equation.

Design

1. Parametric curves

- What is the de Casteljau algorithm
- Prove the correctness of the de Casteljau algorithm
- If a Bezier curve has four control points as $(0,0)$, $(0,2)$, $(8,2)$, $(4,0)$, use the de Casteljau algorithm to compute the curve point when $u=0.5$
- Convert this Bezier curve into a Hermite curve format and a uniform B-spline format.

Hint: Bezier curve $P(u) = \sum_{i=0}^n \binom{n}{i} u^i (1-u)^{n-i} P_i$

Hermite curve:

$$P(u) = \begin{bmatrix} 1 - 3u^2 + 2u^3 & 3u^2 - 2u^3 & u - 2u^2 + u^3 & -u^2 + u^3 \end{bmatrix} \cdot \begin{bmatrix} P_0 \\ P_1 \\ P_0' \\ P_1' \end{bmatrix}$$

B-spline curve: $P(u) = \sum_{i=0}^n P_i N_{i,k}(u)$

$$N_{i,k}(u) = \frac{(u - t_i)N_{i,k-1}(u)}{t_{i+k-1} - t_i} + \frac{(t_{i+k} - u)N_{i+1,k-1}(u)}{t_{i+k} - t_{i+1}}$$

Design

2. Show that if the boundary curves of a bilinear Coons patch are coplanar, the resulting patch is also planar.

Hint: Coons patch is defined as

$$P(u, v) = (1-u)P_0(v) + uP_1(v) + (1-v)Q_0(u) + vQ_1(u) \\ - (1-u)(1-v)P_{0,0} - u(1-v)P_{1,0} - (1-u)vP_{0,1} - uvP_{1,1}$$