Bootstrap Current Optimization in Tokamaks Using Sum-of-Squares Polynomials

A. Gahlawat\textsuperscript{1,2}, E. Witrant\textsuperscript{1}, M. M. Peet\textsuperscript{2}, and M. Alamir\textsuperscript{1}

\textbf{Abstract}—In this paper we present a Lyapunov based feedback design strategy, by employing the sum-of-squares polynomials framework, to maximize bootstrap current in tokamaks. The bootstrap current may play an important role in reducing the external energy input required for tokamak operation. The sum-of-squares polynomials framework allows us to algorithmically construct controllers. Additionally, we provide a heuristic to take into account the control input shape constraints which arise due to limitations on the actuators.

\section{I. INTRODUCTION}

A tokamak is a thermonuclear fusion reactor which uses toroidal and poloidal magnetic fields to heat and compress the plasma (Deuterium and Tritium) to initiate and sustain nuclear fusion reactions [15]. The toroidal and poloidal magnetic coils combined produce a helical magnetic field which confines the plasma. The poloidal field is also generated by the plasma current. The plasma current also contributes to the plasma heating as a consequence of the electrical resistance of the plasma [15].

The main source of plasma current in tokamaks is the current induced in the plasma by the transformer action caused by the central ohmic coil [19]. This current is also known as the induced current. Additional sources of current are the radio-frequency (RF) antennas. The total current provided by these sources thus account for a considerable portion of energy required for tokamak operation. Moreover, because of the transformer action, the tokamak operates as a pulsed device [15]. An additional source of plasma current is the internally generated bootstrap current. The bootstrap current is generated by trapped ions and electrons [19]. Thus, bootstrap current is an automatically generated source of plasma current. An increase in the bootstrap current would lead to a reduced requirement of external current inputs provided by the transformer action and the RF-antennas. This reduced dependence on external current sources would also increase the pulse lengths for which the tokamak can operate. For example, the ultimate goal of the ITER project [7] is to demonstrate the steady state operation of tokamaks. A high value of bootstrap current has been identified as a crucial factor for steady state operation of tokamaks [9], [17].

The bootstrap current density is given by

\[ j_{bs}(x, t) = \frac{p(x, t)}{\partial \psi/\partial x}, \]

where \( p(x, t) \) is a function of the temperature and density profiles of electrons and ions [20], \( \psi(x, t) \) is the poloidal magnetic flux profile, \( x \in [0, 1] \) is the spatial variable and \( t \geq 0 \) is time. Additionally, \( p(x, t) \) is continuous and bounded in both space and time and is positive for \( x \in [0, 1] \) a.e. and \( t \geq 0 \). The dependency of bootstrap current on poloidal flux, temperature and density profiles is explained in [8]. Hence, we aim to maximize \( j_{bs}(x, t) \) by minimizing \( \partial \psi/\partial x \) using the RF-antennas. We achieve this goal by employing a simplified model for the evolution of \( \partial \psi/\partial x \) and by developing Lyapunov functional based conditions. These conditions appear as integral inequalities and to ensure their positivity/negativity we construct controllers such that there integrands are positive/negative. Thus we construct a Lyapunov functional and a controller such that the aforementioned integrands can be represented as sum-of-squares polynomials in certain variables. Additionally, we also provide a heuristic to enforce the spatial shape constraints on the control input. These constraints arise as a consequence of the actuators’ operation limits.

The application of the sum-of-squares framework allows us to algorithmically construct controllers in a computationally effective manner. A few research papers on the application of sum-of-squares polynomials for controller synthesis of infinite-dimensional systems are [14], [5], [12]. An example of application of sum-of-squares polynomials for the control of tokamaks can be found in [6].

The paper is organized as follows: Section II briefly covers the concepts used throughout the paper, Section III provides the main contribution and in Section IV we numerically simulate the controlled system and discuss the results.

\section{II. PRELIMINARIES}

\subsection{A. System model}

We employ the model presented in [20] for the evolution of the poloidal magnetic flux \( \psi \). This model uses the cylindrical approximation and neglects the diamagnetic effect to obtain

\[ \frac{\partial \psi}{\partial t}(x, t) = \frac{\eta_\parallel(x, t)}{\mu_0 a^2} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{x} \frac{\partial \psi}{\partial x} \right) + \eta_\parallel(x, t) R_0 j_{ni}(x, t) \]

with the boundary conditions

\[ \frac{\partial \psi}{\partial x}(0, t) = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial x}(1, t) = -\frac{R_0 \mu_0 I_p(t)}{2\pi}, \]

The authors are with: \textsuperscript{1} UJF / CNRS, Grenoble Image Parole Signal Automatique (GIPSA-lab), UMR 5216, B.P. 46, F-38402 St Martin d’Héres, France; \textsuperscript{2} Cybernetic Systems and Controls Lab (CSCL) in the Department of Mechanical, Materials and Aerospace Engineering, Illinois Institute of Technology, Chicago, IL-60616, USA. E-mail: agahlawa@hawk.iit.edu
where \( I_p(t) \) is the plasma current, \( \eta_j(x, t) \) is the plasma resistivity, \( \mu_0 \) is the permeability of free space, \( a \) is the radius of the last closed magnetic surface, \( R_0 \) is the plasma major radius and \( j_{ni}(x, t) \) is the combined current density resulting from RF and bootstrap sources. Additionally, \( x \in [0, 1] \) is the spatial variable and \( t \geq 0 \) is the time.

To obtain the evolution model of \( Z(x, t) = \psi_z(x, t) \) we differentiate (1) in space to obtain

\[
\frac{\partial Z}{\partial t}(x, t) = \frac{1}{\mu_0 a^2} \frac{\partial}{\partial x} \left( \frac{\eta_j(x, t)}{x} \frac{\partial}{\partial x}(xZ) \right) + R_0 \frac{\partial}{\partial x} \left( \eta_j(x, t) j_{ni}(x, t) \right) \tag{2}
\]

with boundary conditions

\[ Z(0, t) = 0 \quad \text{and} \quad Z(1, t) = -\frac{R_0 \mu_0 I_p(t)}{2\pi}. \tag{3} \]

The non-inductive current density \( j_{ni}(x, t) \) is a sum of the external non-inductive current density \( j_{eni}(x, t) \) and the bootstrap current density \( j_{bs}(x, t) \). Thus, the model can now be represented as

\[
\frac{\partial Z}{\partial t}(x, t) = \frac{1}{\mu_0 a^2} \frac{\partial}{\partial x} \left( \frac{\eta_j(x, t)}{x} \frac{\partial}{\partial x}(xZ) \right) + R_0 \frac{\partial}{\partial x} \left( \eta_j(x, t) j_{eni}(x, t) \right) + R_0 \frac{\partial}{\partial x} \left( \eta_j(x, t) j_{bs}(x, t) \right). \tag{4}
\]

We will also be utilizing the plasma edge value of \( Z_x \). To obtain this value we assume that the total current density \( j_T(x, t) \), defined in [1] as

\[
j_T(x, t) = -\frac{1}{\mu_0 a^2} (xZ_x(x, t) + Z(x, t))
\]

to be zero at the plasma edge. Thus we get

\[ Z_x(1, t) = -Z(1, t). \tag{5} \]

B. Control input

In the system (3), the external non-inductive current density \( j_{eni}(x, t) \) is the control input. The actuators for the control input are the radio frequency (RF) antennas. In particular, two types of RF-antennas are used: lower hybrid current density (LHCD) antennas and electron cyclotron current drive (ECCD) antennas. In this paper we consider the nominal case of having only one LHCD antenna as an actuator.

The authors in [20], using X-ray measurements from Torus Supra, developed empirical scaling laws to approximate the current deposited by the LHCD antennas with Gaussian shapes. In particular

\[
j_{eni}(x, t) = v_{ih}(t)e^{-\left(\frac{(\mu_0 I_p(t) - x)^2}{2\sigma_{ih}(t)}\right)}, \tag{6}
\]

where \( v_{ih}(t) \in [0.1, 2.22 \, \text{MA}], \mu_0 I_p(t) \in [0.14, 0.33] \) and \( \sigma_{ih}(t) \in [0.016, 0.073] \) for all \( t \geq 0 \). Thus, the control input is constrained to be a Gaussian with parameters \( v_{ih}, \mu_0 I_p \) and \( \sigma_{ih} \).

C. Sum-of-squares polynomials

A polynomial \( p(x) \) in variables \( x \in \mathbb{R}^n \) is a sum-of-squares polynomial (SOSP) if there exist polynomials \( p_i(x) \) for \( i \in \{1, \ldots, N\} \) such that

\[
p(x) = \sum_{i=1}^{N} p_i^2(x).
\]

Hence, a SOSP is non-negative by definition. The following theorem provides a necessary and sufficient condition for a polynomial to be SOSP.

**Theorem 1 ([13]):** A polynomial \( p(x), x \in \mathbb{R}^n \) for \( n \in \mathbb{N}_0 \), of degree 2d, \( d \in \mathbb{N} \), is a SOSP if and only if there exists a positive semi-definite and symmetric \( Q \in \mathbb{R}^{d+1} \) such that

\[
p(x) = z(x)^T Q z(x),
\]

where \( z(x) \in \mathbb{R}^{d+1} \) is a vector of monomials in \( x \) up to degree \( d \).

Note that this theorem can easily be generalized for polynomial matrices.

The problem of checking the sign definiteness of a polynomial is NP-hard [2]. However, as a consequence of Theorem 1, the problem of checking for the existence of a SOSP-decomposition for a polynomial is computationally tractable because the condition given by (6) can be represented as linear matrix inequalities (LMIs) [3]. In the presented work, we employ SOSPs to guarantee the negativity/positivity of integrands which in turn imply the positivity/negativity of the respective integrals.

III. MAIN RESULT

In this section we construct a Lyapunov functional which minimizes the gradient of the poloidal magnetic flux \( Z(x, t) \) in order to maximize the bootstrap current density \( j_{bs}(x, t) \). We then proceed by applying integration by parts to obtain an optimization problem with certain polynomials/polynomial matrices as the optimization variables.

**A. Lyapunov functional**

**Theorem 2:** Suppose that there exists a scalar \( \gamma > 0 \) and a strictly positive polynomial \( M : [0, 1] \rightarrow \mathbb{R} \) such that with

\[
V(t) = \int_{0}^{t} Z(x, t)f(x)M^{-1}(x)Z(x, t)dx, \quad f(x) = x^2
\]

the condition

\[
\dot{V}(t) \leq -\frac{\gamma}{\gamma} ||j_{bs}(x, t)||^2_{L_2(0, 1)} - \gamma ||Z(x, t)||^2_{L_2(0, 1)}
\]

holds for all \( t \in [0, T] \) for \( 0 < T < \infty \). Then for all \( t \in [0, T] \) for which

\[
\gamma ||Z(x, t)||^2_{L_2(0, 1)} > \frac{1}{\gamma} ||j_{bs}(x, t)||^2_{L_2(0, 1)},
\]

\( |Z(x, t)||^2_{L_2(0, 1)} \) decreases. Additionally, if for any \( t \in [0, T], |Z(x, t)||^2_{L_2(0, 1)} \leq \frac{1}{\gamma} ||p(x, t)||L_2(0, 1), \quad |Z(x, t)||^2_{L_2(0, 1)} \) is not guaranteed to decrease further. Here, \( L_2(0, 1) \) and \( L_1(0, 1) \) are the spaces of real-valued
square integrable and integrable functions respectively. Additionally \( L^2_{\gamma}(0,1) \) and \( L^p(0,1) \) are the weighted spaces of square integrable and integrable functions, respectively, for weights \( w(x) = M^{-2}(x) \) and \( \hat{w}(x) = M^{-1}(x) \) [10]. The function \( p(x,t) \) is a function of temperature and density profiles and together with \( Z(x,t) \) defines \( j_{bs}(x,t) = p(x,t)/Z(x,t) \).

**Proof:** First note that the weights \( w(x) = M^{-2}(x) \) and \( \hat{w}(x) = M^{-1}(x) \) do not lead to any ill-defined terms as \( M(x) \) is a strictly positive polynomial for all \( x \in [0,1] \). Thus \( M^{-2}(x) \) and \( M^{-1}(x) \) are continuous and bounded functions on \([0,1]\). Suppose that the hypotheses of the theorem hold. Additionally assume that for all \( t \in [T_1,T_2] \), for \( 0 \leq t < T_2 < T \),

\[
\gamma \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} > \frac{1}{\gamma} \|j_{bs}(x,t)\|^2_{L^2_{\gamma}(0,1)}.
\]

Then, since

\[
\dot{V}(t) \leq \frac{1}{\gamma} \|j_{bs}(x,t)\|^2_{L^2_{\gamma}(0,1)} - \gamma \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)},
\]

\[
\dot{V}(t) < 0 \text{ for all } t \in [T_1,T_2].
\]

Integrating in time from \( T_1 \) to \( T_2 \) we get, from (7),

\[
V(T_1) - V(T_2) = \int_0^1 f(x)M^{-1}(x)(Z^2(x,T_1) - Z^2(x,T_2))dx > 0
\]

which implies that

\[
\sup_{x \in [0,1]} f(x) \sup_{x \in [0,1]} M(x) \int_0^1 M^{-2}(x)Z^2(x,T_1)dx
\]

\[
- \sup_{x \in [0,1]} f(x) \sup_{x \in [0,1]} M(x) \int_0^1 M^{-2}(x)Z^2(x,T_2)dx > 0.
\]

Consequently,

\[
\int_0^1 M^{-2}(x)Z(x,T_1)dx > \int_0^1 M^{-2}(x)Z(x,T_2)dx
\]

\[
\Rightarrow \|Z(x,T_1)\|^2_{L^2_{\gamma}(0,1)} > \|Z(x,T_2)\|^2_{L^2_{\gamma}(0,1)}.
\]

Thus, we have shown that for all \( t \) for which

\[
\frac{1}{\gamma} \|j_{bs}(x,t)\|^2_{L^2_{\gamma}(0,1)} - \gamma \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} < 0,
\]

\( \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} \) decreases. Additionally, for all \( t \in [0,T] \) for which

\[
\frac{1}{\gamma} \|j_{bs}(x,t)\|^2_{L^2_{\gamma}(0,1)} - \gamma \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} < 0
\]

\[
\Rightarrow \|j_{bs}(x,t)\|^2_{L^2_{\gamma}(0,1)} < \gamma \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)}
\]

\[
\Rightarrow \|j_{bs}(x,t)\|^2_{L^2_{\gamma}(0,1)} < \gamma \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} < \gamma \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)}
\]

\[
\Rightarrow \|p(x,t)\|_{L^2_{\gamma}(0,1)} \leq M^{-1}(x)\|Z(x,t)\|_{L^2_{\gamma}(0,1)}
\]

\[
\Rightarrow \gamma \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} < 0
\]

where we have used the definition \( j_{bs}(x,t) = p(x,t)/Z(x,t) \) and \( \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} = \|M^{-1}(x)Z(x,t)\|_{L^2_{\gamma}(0,1)} \). Applying the Cauchy-Schwarz inequality on the left hand side

\[
\frac{1}{\gamma} \|p(x,t)\|_{L^2_{\gamma}(0,1)} < \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)}.
\]

Since \( \|p(x,t)\|_{L^2_{\gamma}(0,1)} > 0 \) for all \( t \geq 0 \), we have shown that for all \( t \in [0,T] \) for which

\[
\frac{1}{\gamma} \|j_{bs}(x,t)\|^2_{L^2_{\gamma}(0,1)} - \gamma \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} < 0,
\]

we have that

1) \( \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} \) decreases and
2) \( 0 < \frac{1}{\gamma} \|p(x,t)\|_{L^2_{\gamma}(0,1)} < \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} \).

From these two points we assert that \( \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} \) is minimized for all \( t \in [0,T] \) for which

\[
\frac{1}{\gamma} \|j_{bs}(x,t)\|^2_{L^2_{\gamma}(0,1)} - \gamma \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} < 0.
\]

We will now show that \( \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} \) is not guaranteed to decrease beyond the positive lower bound \( \frac{1}{\gamma} \|p(x,t)\|_{L^2_{\gamma}(0,1)} \). Assume that for some \( t \in [0,T] \),

\[
\|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} = \frac{1}{\gamma} \|p(x,t)\|_{L^2_{\gamma}(0,1)}
\]

\[
\Rightarrow \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} \leq \frac{1}{\gamma} \|p(x,t)\|_{L^2_{\gamma}(0,1)}
\]

\[
\Rightarrow \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} \leq \frac{1}{\gamma} \|j_{bs}(x,t)\|^2_{L^2_{\gamma}(0,1)} Z(x,t)\|_{L^2_{\gamma}(0,1)}
\]

The Cauchy-Schwarz inequality implies that

\[
\|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} \leq \frac{1}{\gamma} \|j_{bs}(x,t)\|^2_{L^2_{\gamma}(0,1)} \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)}
\]

\[
\Rightarrow \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} \leq \frac{1}{\gamma} \|j_{bs}(x,t)\|^2_{L^2_{\gamma}(0,1)}
\]

\[
\Rightarrow \gamma \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} \leq \frac{1}{\gamma} \|j_{bs}(x,t)\|^2_{L^2_{\gamma}(0,1)}
\]

\[
\Rightarrow \frac{1}{\gamma} \|j_{bs}(x,t)\|^2_{L^2_{\gamma}(0,1)} - \gamma \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} \geq 0.
\]

Hence, if for any \( t \in [0,T] \),

\[
\|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} \leq \frac{1}{\gamma} \|p(x,t)\|_{L^2_{\gamma}(0,1)},
\]

\( \dot{V}(t) \) may be non-negative and \( \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} \) may increase towards \( \frac{1}{\gamma} \|j_{bs}(x,t)\|^2_{L^2_{\gamma}(0,1)} \). Note that a higher value of \( \gamma \) would lead to a smaller value of \( \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} \) satisfying the condition

\[
\frac{1}{\gamma} \|j_{bs}(x,t)\|^2_{L^2_{\gamma}(0,1)} - \gamma \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} < 0
\]

which in turn would cause it to decrease further. It would also result in a smaller lower bound \( \frac{1}{\gamma} \|p(x,t)\|_{L^2_{\gamma}(0,1)} \) beyond which \( \|Z(x,t)\|^2_{L^2_{\gamma}(0,1)} \) is not guaranteed to decrease. Thus, we wish to maximize \( \gamma \).
B. Feedback design optimization problem

We now apply integration by parts to the condition in (8) to formulate our optimization problem. We assume that the plasma resistivity can be approximated, as given in [4]:

\[ \eta_i(x, t) = a(t)e^{\lambda(t)x} \] for all \((x, t) \in [0, 1] \times [0, T), \]

where \(0 < a \leq \bar{a} \) and \(0 < \lambda \leq \bar{\lambda} \) are constants.

Theorem 3: Suppose that for a given \( \gamma \) there exist polynomials \( M, R : [0, 1] \to \mathbb{R} \) such that \( M(x) > 0 \) for all \( x \in [0, 1], \Omega(x, \lambda) + \Theta(\gamma) \leq 0 \) for all \((x, \lambda) \in [0, 1] \times [\bar{\lambda}, \bar{\lambda}] \) and \( 2A_4 + 2B_2 + A_2(1) \leq 0, \)

\[
\Omega(x, \lambda) = \begin{bmatrix} 2A_1(x) & 0 & -R_0a\rho a^2f(x) & 0 \\
-2\rho_0a^2f(x) & -R_0a\rho a^2f_z(x) & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\rho a^2}{\rho a^2 + \gamma} \\
\end{bmatrix},
\]

where

\[
A_0(x, \lambda) = 2A_3(x) - \lambda A_2(x) - A_{2,x}(x) + 2B_1(x, \lambda),
\]

\[
A_1(x) = -f(x)M(x),
\]

\[
A_2(x) = -f(x)M(x) - f(x)M_{x}(x) - f_x(x)M_x(x),
\]

\[
A_3(x) = -2M(x) - f_x(x)M_{x}(x),
\]

\[
A_4 = M(1),
\]

\[
B_1(x) = -\frac{f_x(x)R(x)}{2} + \frac{f(x)R_x(x)}{2} + \lambda \frac{f(x)R(x)}{2},
\]

\[
B_2 = \frac{R(1)}{2},
\]

Applying integration by parts on \( V(t) \) and using \( j_{bs}(1, t) = 0 \) we get

\[
\dot{V}_3(t) = \int_0^1 \frac{\eta_i}{\mu_0a^2} Y_s A_1(x)f(x) dx + \int_0^1 \frac{\eta_i}{\mu_0a^2} Y_s A_4(x) \frac{d}{dx}(j_{bs}) dx
\]

Note that we have dropped the spatial and temporal dependencies of the variables for brevity.

We now define \( Z(x, t) = M^{-1}(x) = Y(x, t) \). Hence

\[
\dot{V}_1(t) = \int_0^1 Y f \frac{\partial}{\partial x} \left( \frac{\eta_i}{\mu_0a^2} (xMY) \right) dx,
\]

\[
\dot{V}_2(t) = R_0 \int_0^1 Y f \frac{\partial}{\partial x} (\eta_i j_{bs}) dx
\]

and

\[
\dot{V}_3(t) = R_0 \int_0^1 Y f \frac{\partial}{\partial x} (\eta_i j_{ens}) dx.
\]

Applying integration by parts on \( \dot{V}_1(t) \) we get

\[
\dot{V}_1(t) = \int_0^1 \frac{\eta_i}{\mu_0a^2} (Y_s A_1(x)f(x) + \frac{\partial}{\partial x}(\eta_i KZ)) dx
\]

Applying integration by parts on \( \dot{V}_2(t) \) and using \( j_{bs}(1, t) = 0 \) we get

\[
\dot{V}_2(t) = \int_0^1 \frac{\eta_i}{\mu_0a^2} (Y_f j_{bs} + Y_s (-f)j_{bs}) dx
\]

Using the feedback strategy \( j_{ens}(x, t) = K(x)Z(x, t) / R_0\mu_0a^2 \), we get

\[
\dot{V}_3(t) = \int_0^1 \frac{\eta_i}{\mu_0a^2} (Y_f j_{bs} + Y_s (-f)j_{bs}) dx
\]

Applying integration by parts on \( \dot{V}_3(t) \) we get

\[
\dot{V}_3(t) = \int_0^1 \frac{\eta_i}{\mu_0a^2} (Y_f j_{bs} + Y_s (-f)j_{bs}) dx
\]

Since \( \dot{V}(t) = 2\dot{V}_1(t) + 2\dot{V}_2(t) + 2\dot{V}_3(t) \), using (9), (10) and (11), we obtain

\[
\dot{V}(t) = \int_0^1 \eta_i \frac{\partial}{\partial x} \left( \frac{\eta_i}{\mu_0a^2} (xY) \right) \Omega(x, \lambda) \left[ \begin{array}{c} Y_x \\ Y \end{array} \right] dx
\]

and

\[
\dot{V}_3(t) = R_0 \int_0^1 Y f \frac{\partial}{\partial x} (\eta_i j_{ens}) dx.
\]
Consequently,
\[
\dot{V}(t) - \|j_{bs}\|^2_{L^2(0,1)} + \gamma \|Z\|^2_{L^2(0,1)} = \dot{V}(t) - \|j_{bs}\|^2_{L^2(0,1)} + \gamma \|Y\|^2_{L^2(0,1)} \\
= \int_0^1 \frac{\eta_1}{\mu_0a^2} \left[ \begin{array}{c} Y_x \\ j_{bs} \end{array} \right]^T \Omega(x, \lambda) \left[ \begin{array}{c} Y_x \\ j_{bs} \end{array} \right] dx \\
+ \int_0^1 \left( -\frac{\eta_2}{\gamma} + \gamma Y^2 \right) dx \\
+ \frac{\eta_1(1)}{\mu_0a^2} Y(1,t) (2A_4 + A_2(1) + 2B_2) Y(1,t) \\
+ \frac{\eta_1(1)}{\mu_0a^2} Z_x(1,t) Y(1,t) \\
= \frac{\eta_1}{\mu_0a^2} \left[ \begin{array}{c} Y_x \\ j_{bs} \end{array} \right]^T \Omega(x, \lambda) \left[ \begin{array}{c} Y_x \\ j_{bs} \end{array} \right] dx \\
+ \int_0^1 \frac{\eta_1}{\mu_0a^2} \left( \frac{\mu_0a^2Y^2}{\eta \gamma} + \frac{\mu_0a^2Y^2}{\eta} \right) dx \\
+ \frac{\eta_1(1)}{\mu_0a^2} Y(1,t) (2A_4 + A_2(1) + 2B_2) Y(1,t) \\
+ \frac{\eta_1(1)}{\mu_0a^2} Z_x(1,t) Y(1,t). \tag{12}
\]

Since \( \eta_1(x, t) = a(t) e^{\lambda x} \), \( a \leq \eta_1(x, t) \leq ae^\lambda \) for all \( (x, t) \in [0, 1] \times [0, T] \). Hence,
\[
\left[ \begin{array}{c} Y_x \\ j_{bs} \end{array} \right]^T \Omega(x, \lambda) \left[ \begin{array}{c} Y_x \\ j_{bs} \end{array} \right] = \frac{\mu_0a^2Y^2}{\eta_1} + \frac{\mu_0a^2Y^2}{\eta} \\
\leq \frac{\mu_0a^2Y^2}{\eta_1} + \frac{\mu_0a^2Y^2}{\eta} = \left( \frac{\mu_0a^2Y^2}{\eta_1} + \frac{\mu_0a^2Y^2}{\eta} \right) \left( \Omega(x, \lambda) + \Theta \right) \\
= \left[ \begin{array}{c} Y_x \\ j_{bs} \end{array} \right]^T \left( \Omega(x, \lambda) + \Theta \right) \left[ \begin{array}{c} Y_x \\ j_{bs} \end{array} \right].
\]

Since \( \Omega(x, \lambda) + \Theta \leq 0 \) for all \( (x, \lambda) \in [0, 1] \times [\lambda, \bar{\lambda}] \), we conclude that
\[
\int_0^1 \frac{\eta_1}{\mu_0a^2} \left[ \begin{array}{c} Y_x \\ j_{bs} \end{array} \right]^T \Omega(x, \lambda) \left[ \begin{array}{c} Y_x \\ j_{bs} \end{array} \right] dx \\
+ \int_0^1 \frac{\eta_1}{\mu_0a^2} \left( \frac{\mu_0a^2Y^2}{\eta_1} + \frac{\mu_0a^2Y^2}{\eta} \right) dx \leq 0 \tag{13}
\]
for all \( t \geq 0 \). Similarly, since from the theorem hypothesis \( 2A_4 + A_2(1) + 2B_2 \leq 0 \),
\[
\frac{\eta_1(1)}{\mu_0a^2} Y(1,t) (2A_4 + A_2(1) + 2B_2) Y(1,t) \leq 0 \tag{14}
\]
Finally, from the boundary conditions given in (2) and (4) coupled with the definition of \( Y(x, t) \), it is straightforward to observe that
\[
\frac{\eta_1(1)}{\mu_0a^2} Z_x(1,t) Y(1,t) \leq 0 \tag{15}
\]
Combining equations (12), (13), (14) and (15) completes the proof.

C. Shape constraints

We would like to comment that a state-feedback control is possible to implement because of the availability of the on-line estimate of the state \( Z(x, t) \) as detailed in [11].

We now implement shape constraints on the control input \( j_{ens}(x,t) \). To ensure that \( j_{ens}(x,t) \) resembles Gaussians of feasible parameters \( v_0, \mu_0, \) and \( \sigma_0 \), we add an additional constraint of the form
\[
g_1(x) \leq j_{ens}(x,t) = \frac{K(x)}{R_0\mu_0a^2} Z(x,t) \leq g_2(x),
\]
where \( g_1(x) < g_2(x) \), for all \( x \in [0, 1] \), are polynomial approximations of two selected feasible Gaussians. Since both \( K(x) \) and \( Z(x,t) \) are continuous, the control input is a continuous function bounded by the shape of the constraint envelope defined by \( g_1(x) \) and \( g_2(x) \). Additionally, we assume that
\[
Z(x,t) = \alpha(x)Z_1(x) + (1 - \alpha(x))Z_2(x)
\]
for all \( (x, \alpha) \in [0, 1] \times [0, 1] \). Since this is a heuristic, we try different constraint envelopes to choose \( g_1(x) \) and \( g_2(x) \).

We conclude this section by stating the optimization problem that is finally solved:
Maximize \( \gamma > 0 \) such that for given polynomials \( Z_1(x), Z_2(x), g_1(x) \) and \( g_2(x) \) there exist polynomials \( M(x) \) and \( R(x) \) satisfying:
1) \( M(x) > 0 \) for all \( x \in [0, 1], \)
2) \( \Omega(x, \lambda) + \Theta \leq 0 \) for all \( (x, \lambda) \in [0, 1] \times [\lambda, \bar{\lambda}], \)
3) \( 2A_4 + 2B_2 + A_2(1) \leq 0 \) and
4) \( R_0\mu_0a^2 M(x) g_1(x) \leq R(x) (\alpha Z_1(x) + (1 - \alpha) Z_2(x)) \leq R_0\mu_0a^2 M(x) g_2(x) \)
for all \( (x, \alpha) \in [0, 1] \times [0, 1], \)
where \( \Omega(x, \lambda), \Theta, A_4, A_2(x) \) and \( B_2 \) are defined in the statement of Theorem 3.

We solve the optimization problem using SOSTOOLS [16] which is a toolbox for MATLAB®. The search for the maximum \( \gamma \) is performed using the bisection method. We solve this problem for the Tore Supra tokamak for which \( R_0 = 2.38m \) and \( a = 0.38m \). Moreover, the plasma resistivity is defined as \( \eta_1(x, t) = a(t) e^{\lambda x} \) where \( a(t) \in [0.0093, 0.0121] \) and \( \lambda(t) \in [4, 7.3] \) for all \( t \geq 0 \). These values were obtained from the data for shot TS 35109.
Fig. 1: Constraint envelope and $\frac{K(x)}{R_{\text{loop}}} (\alpha Z_1(x) + (1 - \alpha)Z_2(x))$ for $\alpha \in [0, 1]$

Fig. 2: Evolution of closed loop and open loop $\|j_{bs}(x, t)\|_{L_2(0, 1)}$

IV. SIMULATION

We obtain a maximum value of $\gamma = 10^4$ as the solution for the optimization problem for Tore Supra. The feasible polynomials $M(x)$ and $R(x)$ obtained for this value of $\gamma$ are of degree 12 in the spatial variable $x$. We simulate the closed-loop system on the simulator developed in [20]. The following figures provide the simulation results.

Figure 1 shows the constraint envelope as well as $\frac{K(x)}{R_{\text{loop}}} (\alpha Z_1(x) + (1 - \alpha)Z_2(x))$ for $\alpha \in [0, 1]$, where $K(x) = R(x)/M(x)$.

Figure 2 shows the comparison between the time evolution of the spatial $L_2$-norm of open-loop and closed-loop $j_{bs}(x, t)$. With the synthesized controllers we are able to obtain a percentage increase of $\approx 90\%$.

Figures 3a and 3b illustrate the time evolution of the controlled state $Z(x, t)$ and $j_{bs}(x, t)$ respectively. The effect of the controller can be observed by comparing the contours before and after $t = 12s$.

Finally, to analyse the control input shapes, we fit feasible Gaussians to control inputs at various times as shown in Figure 4. We observe that the control input approximates the shape of feasible Gaussians satisfactorily for roughly 70\% of the spatial domain. However, the control input departs from the Gaussian shapes as $x \to 0$. This is due to the controller having the form

$$j_{\text{ens}}(x, t) = \frac{K(x)}{R_{\text{loop}}} Z(x, t)$$

and the boundary condition $Z(0, t) = 0$. Note that the Gaussian approximation of the LHCD current deposit is obtained from hard X-ray measurements and, as stated in [20], a large uncertainty remains concerning the actual deposit close to the plasma center ($x = 0$). Furthermore, the use of RF-antennas generating a sharper deposit (such as ECCD) would satisfy the zero value condition of the control input at the plasma center.

V. CONCLUSIONS AND FUTURE RESEARCH

In the presented work we investigate the applicability of the sum-of-squares polynomials framework for the control of tokamak plasmas. In particular, we devise a strategy to maximize the bootstrap current density in the plasma while taking into account the time varying plasma resistivity and the non-linearity present in the model because of the bootstrap current. Our method also provides a heuristic to constrain the shape of the control inputs. The simulation results illustrate the effectiveness of the proposed method. Additionally, this method is computationally effective method for synthesizing controllers numerically, provided that the control gain $K(x)$ can be computed off-line. This can be considered a weak condition considering the steady-state operation objectives or
by taking sufficiently large bounds on the admissible steady state profiles.

However, additional research has to be performed before this method can provide controllers which can be tested under realistic operating scenarios. An additional loop is required to infer the antennas’ engineering parameters (power, refractive index for LHCD and orientation for ECCD) that minimize the difference between the desired and effective current deposit. We will also be investigating the use of bootstrap current density to develop feedback laws since online measurement of the pressure profiles, in addition to the gradient of the poloidal magnetic flux profiles, are available. We would eventually be validating the controllers on METIS [18], an advanced simulator for toroidal plasmas. We would also generalize this method to include multiple LHCD and ECCD antennas.

REFERENCES