A Converse Sum of Squares Lyapunov Result with a Degree Bound

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Abstract

In this paper, we show that local exponential stability of a polynomial vector field implies the existence of a Lyapunov function which is a sum-of-squares of polynomials. In particular, the main result states that if a system is exponentially stable system on a bounded nonempty set, then there exists an SOS Lyapunov function which is exponentially decreasing on that bounded set. The proof is constructive and uses the Picard iteration. A bound on the degree of this converse Lyapunov function is also given. This result implies that semidefinite programming can be used to answer the question of local stability of a polynomial vector field with a bound on complexity.

Index Terms

I. Introduction

Computational methods are extensively used in the analysis of dynamical systems; a particular example is semidefinite programming for linear control problems. At the same time, many nonlinear and infinite-dimensional problems can be formulated as polynomial non-negativity conditions. The ability to optimize over the set of positive polynomials using the sum-of-squares relaxation has opened up new ways for addressing nonlinear control problems, in much the same way Linear Matrix Inequalities are used to address analysis questions for linear finite-dimensional systems. However, there remain several open questions about how these methods can be used to search for Lyapunov functions for nonlinear systems. For references on early work on optimization of polynomials, see [1], [2], and [3]. For more

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recent work see [4] and [5]. Today, there exist a number of software packages for optimization over positive polynomials, e.g. SOSTOOLS [6] and GloptiPoly [7].

In this paper we focus on the use of sum-of-squares Lyapunov functions for the analysis of nonlinear systems - we do not detail the process of actually computing sum-of-squares Lyapunov functions, see [3], [8], [9] and [10]. Instead, we concentrate on the local stability of the zero equilibrium of

$$\dot{x}(t) = f(x(t)),$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is polynomial. In particular, in this paper we address the question of whether an exponentially stable nonlinear system will have a sum-of-squares Lyapunov function which establishes this property. In previous work [11], we were able to show that local stability on a bounded region implies the existence of a exponentially decreasing polynomial Lyapunov function on that set.

The results of this paper are in fact related to the question of whether stability of a system implies the existence of a Lyapunov function and what the properties of that function are. Particularly relevant work includes research on continuity properties, See e.g. [12], [13] and [14] and the overview in [15]. Infinitely-differentiable functions were explored in the work [16], [17]. Other innovative results are found in [18] and [19]. The books [20] and [21] treat further converse theorems of Lyapunov.

There are two technical contributions in this paper to the development of converse Lyapunov theory. Unlike the work in [11], this paper is more closely tied to systems theory in that we approximate the solution map rather than the Lyapunov function directly. This approximate solution map is used to develop a converse Lyapunov function. The first key insight is to note that, due to the structure of this converse function, if the approximation to the solution map is polynomial, then the Lyapunov function will be a sum of squares. This improvement is important because it is possible to optimize over sum-of-squares polynomials, while it is not currently possible to optimize over positive polynomials.

The second key insight is to use the Picard iteration to approximate the solution map instead of standard polynomial approximations such as Bernstein polynomials. The reason is that the Picard iteration retains several key features of the solution map. It is well-known that the Picard iteration is not ideally suited for approximation of the solution map in a general context, as it only converges on a short interval. However, when approximating the solution map for a stable system, we show how the Picard iteration can be extended indefinitely to create polynomial approximations of the solution map.

The main result of the paper is stated and proven in Section VII. The sections leading to Section VII present a series of lemmas that are used in the proof of the main theorem. In Section III we prove that the Picard iteration satisfies a shifting property; in Section IV we show that the Picard iteration is contractive on a certain metric space; and in Section V we propose a new way of extending the Picard iteration. In Section VI we show that the Picard iteration approximately retains the differentiability properties of the solution map, before we prove the main result. The paper is concluded in Section IX.

II. Notation and Background

The core concept we use in this paper is the Picard iteration. We use this to construct an approximation to the solution map and then use the approximate solution map to construct the Lyapunov function. Construction of the Lyapunov function will be discussed in more depth later on. However, at this point we review the Picard iteration: a standard method for proving the existence of solutions.

Denote the Euclidean ball centered at 0 of radius r by B_r. Consider an ordinary differential equation of the form

$$\dot{x}(t) = f(x(t)), \qquad x(a) = x_0, \qquad f(0) = 0.$$

where $x \in \mathbb{R}^n$ and f satisfies appropriate smoothness properties for local existence and uniqueness of solutions. The solution map is a function ϕ which satisfies

$$\frac{\partial}{\partial t}\phi(t,a,x) = f(\phi(t,a,x)) \qquad \text{and} \qquad \phi(a,a,x) = x.$$

Of course, for a time-invariant system, the solution map could also be expressed as $\phi(s,t)$. However, we do not make this change in order to preserve certain properties of the solution map which we will need to prove the main theorem.

Definition 1: Let X be a metric space. A mapping $F : X \to X$ is *contractive* with coefficient $d \in [0,1)$ if

$$\|Fx - Fy\| \leq d\,\|x-y\| \qquad x,y \in X.$$

The following is a *Fixed-Point* Theorem.

Theorem 2 (Contraction Mapping Principle): Let X be a complete metric space and let $F : X \to X$ be a contraction with coefficient d. Then there exists a unique $a \in X$ such that

$$Fa = a.$$

Furthermore

$$\left\|F^k x_0 - a\right\| \leq d^k \left\|x_0 - a\right\|.$$

To apply these results to the existence of the solution map, we use the Picard iteration.

Definition 3: For given x, T and r, define the metric space

$$X := \left\{ z(t,a,x) : \ \sup_{t\in[a,a+T]} \|z(t,a,x)\| \leq 2r, \ z \text{ is continuously differentiable.} \right\} \quad (1)$$

with norm $\|z\| = \sup\limits_{t\in[a,a+T]} \|z(t,a,x)\|$.

Finally, define the *Picard iteration*,

$$(Pz)(t,a,x) \triangleq x + \int_a^t f(z(s,a,x))ds.$$

III. Picard Shift Invariance Lemma

The first result is a technical Lemma showing that the Picard iteration satisfies the time-invariance property of the solution map.

Lemma 4: Let $z(s,t,x) = 0$. For a time-invariant system, the Picard iteration satisfies

$$(P^k z)(s,t,x) = (P^k z)(s-a,t-a,x)$$

Proof: The proof is by induction. At the first iteration,

$$(Pz)(s,t,x) = (Pz)(s-a,t-a,x) = x.$$

Suppose that $(P^k z)(s,t,x) = P^k z(s-a,t-a,x)$. Then by shift-invariance of P^k,

$$\begin{aligned}(P^{k+1}z)(s,t,x) &= x + \int_t^s f((P^k z)(\omega,t,x))d\omega = x + \int_{t-a}^{s-a} f((P^k z)(\omega+a,t,x))d\omega \\ &= x + \int_{t-a}^{s-a} f((P^k z)(\omega,t-a,x))d\omega = (P^{k+1}z)(s-a,t-a,x).\end{aligned}$$

Therefore, the Lemma holds by induction. ■

IV. Picard Iteration

We begin this section by showing that for any radius r, there exists a T such that the Picard iteration is contractive on X for any $x \in B_r$.

Lemma 5: Given $r > 0$, let $T < \min\{\frac{r}{Q}, \frac{1}{L}\}$ where f has Lipschitz factor L on B_{2r} and $Q = \sup_{x\in B_{2r}} f(x)$. Then for any $x \in B_r$, $P : X \to X$ and there exists some $\phi \in X$ such that for $t \in [a,a+T]$,

$$\frac{d}{dt}\phi(t,a,x) = f(\phi(t,a,x)), \qquad \phi(a,a,x) = x$$

and for any $z \in X$,

$$\left\|\phi - P^k z\right\| \le (TL)^k \|\phi - z\|.$$

Proof: We first show $P : X \to X$. If $z \in X$, then for $x \in B_r$, $\|z(s,t,x)\| \le 2r$ and so

$$\begin{aligned}\sup_{t\in[a,a+T]} \|(Pz)(t,a,x)\| &= \sup_{t\in[a,a+T]} \left\| x + \int_a^t f(z(s,a,x)) \right\| ds \\ &\le \|x\| + \int_a^{a+T} \|f(z(s,a,x))\| ds \le r + TQ < 2r\end{aligned}$$

Thus we conclude that $Pz \in X$. Furthermore, for $z_1, z_2 \in X$,

$$\begin{aligned}\|Pz_1 - Pz_2\| &= \sup_{t\in[a,a+T]} \left\| \int_a^t (f(z_1(s,a,x)) - f(z_2(s,a,x))) ds \right\| \\ &\le \int_a^{a+T} \|f(z_1(s,a,x)) - f(z_2(s,a,x))\| ds \\ &\le TL \sup_{s\in[a,a+T]} \|z_1(s,a,x) - z_2(s,a,x)\| = TL \|z_1 - z_2\|\end{aligned}$$

Therefore, by the contraction mapping theorem, the Picard iteration converges on $[0,T]$ with convergence rate $(TL)^k$. ■

V. Picard Extension Convergence Lemma

In this section we propose a new way of extending the Picard iteration. We use the final value of the previous Picard iteration as the initial condition for a new round of Picard iteration – see Figure 1 for an illustration. This is done to achieve convergence on an arbitrary interval while maintaining the polynomial nature of the approximation on several subintervals.

Definition 6: Suppose that the solution map ϕ exists on $(s-t) \in [0,\infty]$ and $\|\phi(s,t,x)\| \le K\|x\|$ for any $x \in B_r$. Suppose that f has Lipschitz factor L on B_{4Kr} and is bounded on B_{4Kr} with bound Q. Given $T < \min\{\frac{2Kr}{Q}, \frac{1}{L}\}$, let $z = 0$ and define

$$G_0^k(s,t,x) := (P^k z)(s,t,x)$$

and for $i > 0$, define the functions G_i recursively as

$$G_{i+1}^k(s,t,x) := (P^k z)(s,t,G_i^k(T,0,x)).$$

Define the concatenation of the G_i^k as

$$G^k(s,t,x) := G_i(s-iT,t,x) \quad \forall \quad s \in [t+iT, t+iT+T] \quad \text{and } i = 1,\cdots,\infty.$$

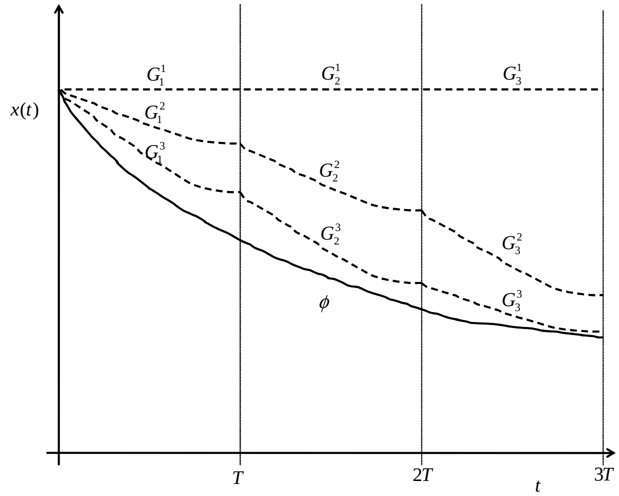

Fig. 1. An illustration of the solution map ϕ and the functions G_i^k which are used in the Extended Picard Iteration and the Lyapunov construction.

If f is polynomial, then the G_i^k are polynomials for any i,k and G^k is continuously differentiable for any k. The following lemma provides several properties for the functions G^k.

Lemma 7: Given $\delta > 0$, suppose that the solution map $\phi(s,t,x)$ exists on $(s-t) \in [0,\delta]$ and $x \in B_r$ and $\|\phi(s,t,x)\| \le K\|x\|$ for any $x \in B_r$. Suppose that f is Lipschitz on B_{4Kr} with factor L and bounded with bound Q. Choose $T < \min\{\frac{2Kr}{Q}, \frac{1}{L}\}$ and integer $N > \delta/T$. Then let G^k and G_i^k be defined as above.

Define the function

$$c(k) = \sum_{i=1}^{N} \left(e^{TL} + K^2(TL)^k\right)^i K^2(TL)^k.$$

Then for any k such that $c(k) < K$, $G^k \in Y$ where

$$Y := \left\{ z(t,a,x) : \begin{array}{c} \sup_{\substack{t\in[a,a+\delta] \\ x\in B_r}} \|z(t,a,x)\| \le r, \ z \text{ is continuously differentiable and} \\ G^k(s,t,x) = G^k(s-a,t-a,x). \end{array} \right\}. \quad (2)$$

Furthermore

$$\left\| G^k(s,0,x) - \phi(s,0,x) \right\| \le c(k)\|x\|.$$

Proof: Define the convergence rate $d = TL < 1$. Then the conditions of Lemma 5 are satisfied using $r' = 2Kr$. Thus for any $x \in B_{2Kr}$, P^k converges to ϕ on $[0,T]$. Then

$$\sup_{s\in[0,T]} \left\| G_0^k(s,0,x) - \phi(s,0,x) \right\| = \sup_{s\in[0,T]} \left\| P^k(s,0,x) - \phi(s,0,x) \right\| \le d^k \sup_{s\in[0,T]} \|\phi(s,0,x)\| \le Kd^k\|x\|.$$

Thus G^k converges to ϕ on the interval $[0,T]$. Define

$$c_i(k) = \sum_{j=1}^{i} (e^d + K^2 d^k)^j K^2 d^k.$$

and suppose that $\left\|G^k - \phi\right\| \leq c_{i-1}(k)\,\|x\|$ on interval $[iT-T, iT]$. Then

$$\begin{aligned}
\sup_{s\in[iT,iT+T]} \left\|G^k(s,0,x) - \phi(s,0,x)\right\| &= \sup_{s\in[iT,iT+T]} \left\|G_i^k(s-iT,0,x) - \phi(s,0,x)\right\| \\
&= \sup_{s\in[iT,iT+T]} \left\|P^k(s-iT,0,G_{i-1}^k(T,0,x)) - \phi(s-iT,0,\phi(iT,0,x))\right\| \\
&\leq \sup_{s\in[iT,iT+T]} \left\|P^k(s-iT,0,G_{i-1}^k(T,0,x)) - \phi(s-iT,0,G_{i-1}^k(T,0,x))\right\| \\
&\quad + \sup_{s\in[iT,iT+T]} \left\|\phi(s-iT,0,G_{i-1}^k(T,0,x)) - \phi(s-iT,0,\phi(iT,0,x))\right\|
\end{aligned}$$

We treat these final two terms separately. First note that

$$\begin{aligned}
\left\|G_{i-1}^k(T,0,x)\right\| &\leq \|\phi(iT,0,x)\| + \left\|\phi(iT,0,x) - G_{i-1}^k(T,0,x)\right\| \leq K\,\|x\| + c_{i-1}(k)\,\|x\| \\
&\leq (K + c_{i-1}(k))\,\|x\|\,.
\end{aligned}$$

Since $c_{i-1}(k) \leq c(k) < K$ and $x \in B_r$, $\left\|G_{i-1}^k(T,0,x)\right\| \leq (K + Kc_{i-1}(k))\,\|x\| \leq 2Kr$. Hence

$$\begin{aligned}
&\sup_{s\in[iT,iT+T]} \left\|P^k(s-iT,0,G_{i-1}^k(T,0,x)) - \phi(s-iT,0,G_{i-1}^k(T,0,x))\right\| \\
&\leq \sup_{s\in[iT,iT+T]} d^k \left\|\phi(s-iT,0,G_{i-1}^k(T,0,x))\right\| \leq Kd^k \left\|G_{i-1}^k(T,0,x)\right\| \\
&\leq Kd^k(K + c_{i-1}(k))\,\|x\|\,.
\end{aligned}$$

Now, if $x \in B_r$, $\|\phi(s,0,x)\| \leq Kr$ and since $\left\|G_{i-1}^k(T,0,x)\right\| \leq 2Kr$ and f is Lipschitz on B_{4Kr}, it is well-known that

$$\begin{aligned}
&\sup_{s\in[iT,iT+T]} \left\|\phi(s-iT,0,G_{i-1}^k(T,0,x)) - \phi(s-iT,0,\phi(iT,0,x))\right\| \\
&\leq \sup_{s\in[iT,iT+T]} e^{L(s-iT)} \left\|G_{i-1}^k(T,0,x) - \phi(iT,0,x)\right\| \leq e^{TL} c_{i-1}(k)\,\|x\|
\end{aligned}$$

Combining, we conclude that

$$\begin{aligned}
\sup_{s\in[iT,iT+T]} \left\|G_i^k(s-iT,0,x) - \phi(s,0,x)\right\| &\leq e^{TL} c_{i-1}(k)\,\|x\| + Kd^k(K + c_{i-1}(k))\,\|x\| \\
&= ((e^d + Kd^k)c_{i-1}(k) + K^2 d^k)\,\|x\| = c_i(k)\,\|x\|\,.
\end{aligned}$$

The inequality holds for $i=0$ by assumption. By induction, we conclude that

$$\left\|G^k(s,0,x)-\phi(s,0,x)\right\| \leq c(k)\,\|x\|\,.$$

To show $G^k(s,t,x)=G^k(s-a,t-a,x)$, recall

$$G^k(s,t,x) := G_i(s-iT,t,x) \quad \forall \quad s\in[t+iT,t+iT+T] \quad \text{and } i=1,\cdots,\infty.$$

By definition then, $G^k(s-a,t-a,x)=G_i(s-iT-a,t-a,x)$ for $s\in[t+iT,t+iT+T]$. Now since $G^k_{i+1}(s,t,x)=P^k(s,t,G^k_i(T,0,x))$, we have

$$G_i(s-iT-a,t-a,x)=P^k(s-a,t-a,G^k_{i-1}(T,0,x))=P^k(s,t,G^k_{i-1}(T,0,x))=G_i(s-iT,t,x)$$

for $s\in[t+iT,t+iT+T]$. Therefore, $G^k(s-a,t-a,x)=G^k_i(s-iT,t,x)=G^k(s,t,x)$ for $s\in[t+iT,t+iT+T]$, $i=1,\ldots,N$. This implies that $G^k(s-a,t-a,x)=G^k(s,t,x)$ for all s. ■

VI. Derivative Inequality Lemma

In this critical lemma, we show that the Picard iteration approximately retains the differentiability properties of the solution map. The proof is based on induction and is inspired by an approach in [22]. This lemma is then adapted to the extended Picard iteration introduced in the previous section.

Lemma 8: Suppose that the conditions of Lemma 5 are satisfied. Then for any $x\in B_r$ and any $k\geq 0$,

$$\sup_{t\in[a,a+T]}\left\|\frac{\partial}{\partial a}(P^k z)(t,a,x)+\frac{\partial}{\partial x}(P^k z)(t,a,x)^T f(x)\right\| \leq \frac{(TL)^k}{T}\,\|x\|$$

Proof: Begin with the identity for $k\geq 1$

$$(P^k z)(t,a,x)=x+\int_a^t f((P^{k-1}z)(s,a,x))ds.$$

Then

$$\begin{aligned}\frac{\partial}{\partial a}(P^k z)(t,a,x) &= -f((P^{k-1}z)(a,a,x))+\int_a^t \nabla f((P^{k-1}z)(s,a,x))^T\frac{\partial}{\partial a}(P^{k-1}z)(s,a,x)ds\\ &= -f(x)+\int_a^t \nabla f((P^{k-1}z)(s,a,x))^T\frac{\partial}{\partial a}(P^{k-1}z)(s,a,x)ds,\end{aligned}$$

and

$$\frac{\partial}{\partial x}(P^k z)(t,a,x)=I+\int_a^t \nabla f((P^{k-1}z)(s,a,x))^T\frac{\partial}{\partial x}(P^{k-1}z)(s,a,x)ds.$$

Now define for $k \geq 1$,

$$y_k(t,a,x) := \frac{\partial}{\partial a}(P^k z)(t,a,x) + \frac{\partial}{\partial x}(P^k z)(t,a,x)^T f(x).$$

For $k \geq 2$, we have

$$\begin{aligned}
y_k(t,a,x) &:= \frac{\partial}{\partial a}(P^k z)(t,a,x) + \frac{\partial}{\partial x}(P^k z)(t,a,x)^T f(x) \\
&= \int_a^t \nabla f((P^{k-1}z)(s,a,x))^T \frac{\partial}{\partial a}(P^{k-1}z)(s,a,x)ds \\
&\quad + \int_a^t \nabla f((P^{k-1}z)(s,a,x))^T \frac{\partial}{\partial x}(P^{k-1}z)(s,a,x) f(x)ds \\
&= \int_a^t \nabla f((P^{k-1}z)(s,a,x))^T \left[\frac{\partial}{\partial a}(P^{k-1}z)(s,a,x) + \frac{\partial}{\partial x}(P^{k-1}z)(s,a,x) f(x)\right] ds \\
&= \int_a^t \nabla f((P^{k-1}z)(s,a,x))^T y_{k-1}(s,a,x)ds.
\end{aligned}$$

This means that since $(P^{k-1}z)(t,a,x) \in B_{2r}$, by induction

$$\begin{aligned}
\sup_{[a,a+T]} \|y_k(t)\| &\leq T \sup_{t\in[a,a+T]} \left\|\nabla f((P^{k-1}z)(t,a,x))\right\| \sup_{t\in[a,a+T]} \|y_{k-1}(t,a,x)\| \\
&\leq TL \sup_{t\in[a,a+T]} \|y_{k-1}(t,a,x)\| \leq (TL)^{(k-1)} \sup_{t\in[a,a+T]} \|y_1(t,a,x)\|
\end{aligned}$$

For $k = 1$, $(Pz)(t,a,x) = x$, so $y_1(t) = f(x)$ and $\sup_{[a,a+T]} \|y_1(t)\| \leq L\|x\|$. Thus

$$\sup_{[a,a+T]} \|y_k(t)\| \leq \frac{(TL)^k}{T} \|x\|.$$

■

We now adapt this lemma to the extended Picard iteration.

Lemma 9: Suppose that the conditions of Lemma 7 are satisfied. Then for any $x \in B_r$,

$$\sup_{t\in[a,a+T]} \left\| \frac{\partial}{\partial a} G^k(t,a,x) + \frac{\partial}{\partial x} G^k(t,a,x)^T f(x) \right\| \leq \frac{(TL)^k}{T}(K + c(k))\|x\|$$

Proof: Recall that

$$G^k(s,t,x) := G_i(s - iT, t, x) \quad \forall \quad s \in [t + iT, t + iT + T] \quad \text{and } i = 1, \cdots, \infty.$$

and $G_{i+1}^k(s,t,x) = P^k(s,t,G_i^k(T,0,x))$. For $t \in [a+iT, a+iT+T]$,

$$\begin{aligned}\left\| \frac{\partial}{\partial a} G^k(t,a,x) + \frac{\partial}{\partial x} G^k(t,a,x)^T f(x) \right\| &= \left\| \frac{\partial}{\partial a} G_i^k(t-iT,a,x) + \frac{\partial}{\partial x} G^k(t-iT,a,x)^T f(x) \right\| \\ &= \left\| \frac{\partial}{\partial a} P^k(t-iT,a,G_i^k(T,0,x)) + \frac{\partial}{\partial x} P^k(t-iT,a,G_i^k(T,0,x))^T f(x) \right\| \\ &\leq \frac{(TL)^k}{T} \left\| G_i^k(T,0,x) \right\|\end{aligned}$$

As was shown in the proof of Lemma 7, $\left\| G_i^k(T,0,x) \right\| \leq (K + c_i(k)) \|x\|$. Thus

$$\left\| \frac{\partial}{\partial a} G^k(t,a,x) + \frac{\partial}{\partial x} G^k(t,a,x)^T f(x) \right\| \leq \frac{(TL)^k}{T} (K + c_i(k)) \|x\|$$

Since the c_i are non-decreasing,

$$\sup_{t \in [a,a+\delta]} \left\| \frac{\partial}{\partial a} G^k(t,a,x) + \frac{\partial}{\partial x} G^k(t,a,x)^T f(x) \right\| \leq \frac{(TL)^k}{T} (K + c(k)) \|x\| .$$

■

VII. Main Result - A Converse SOS Lyapunov Function

In this section, we combine the previous results in a relatively straightforward manner to obtain a converse Lyapunov function which is also a sum-of-squares polynomial. Specifically, we use a standard form of converse Lyapunov function and substitute our extended Picard iteration for the solution map. Consider the system

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0. \tag{3}$$

Theorem 10: Suppose that f is polynomial of degree q and that system (3) is exponentially stable with

$$\|x(t)\| \leq K \|x(0)\| e^{-\lambda t}$$

for some $\lambda > 0$, $K \geq 1$ and for any $x(0) \in M$, where M is a bounded nonempty region of radius r. Then there exist $\alpha, \beta, \gamma > 0$ and a sum-of-squares polynomial $V(x)$ such that for any $x \in M$,

$$\alpha \|x\|^2 \leq V(x) \leq \beta \|x\|^2$$
$$\nabla V(x)^T f(x) \leq -\gamma \|x\|^2 .$$

Further, the degree of V will be less than $q^{2(Nk-1)}$, where k is the minimum integer such that

$c(k) < K$, and

$$c(k)^2 + \frac{\log 2K^2}{2\lambda} KL(1+c(k))(K+c(k)) < \frac{1}{2},$$
$$c(k)^2 < \frac{\lambda}{KL\log 2K^2}(1-(2K^2)^{-\frac{L}{\lambda}}).$$

where $c(k)$ is defined as

$$c(k) = \sum_{i=1}^{N} \left(e^{TL} + K^2(TL)^k\right)^i K^2(TL)^k.$$

where T and N are chosen such that $NT > \frac{\log 2K^2}{2\lambda}$ and $T < \min\{\frac{2Kr}{Q}, \frac{1}{L}\}$ where L is a Lipschitz bound for f on B_{4Kr}.

Proof: Define $\delta = \frac{\log 2K^2}{2\lambda}$ and $d = TL$. We note that since stability implies $f(0) = 0$, f is bounded on B_r with bound $Q = Lr$. Thus since $\frac{2Kr}{Q} = \frac{2K}{L} > \frac{1}{L}$, the conditions of Lemma 5 are satisfied.By Lemma 5, the Picard iteration converges on $[0,T]$ for any $x \in B_{2Kr}$ with rate d^k. Define G^k as in Lemma 7. By Lemma 7, if k is defined as above, $\left\|G^k(s,0,x) - \phi(s,0,x)\right\| \leq c(k)\|\phi(s,0,x)\|$ on $s \in [0,\delta]$ and $x \in B_r$. We propose the following Lyapunov function, indexed by k.

$$V_k(x) := \int_0^\delta G^k(s,0,x)^T G^k(s,0,x) ds$$

The proof is divided into four parts:

a) Upper and Lower Bounded: To prove that V_k is a valid Lyapunov function, first consider upper boundedness. If $x \in B$ and $s \in [0,\delta]$. Then

$$\begin{aligned}\left\|G^k(s,0,x)\right\|^2 &= \left\|\phi(s,0,x) + \left[G^k(s,0,x)^T - \phi(s,0,x)\right]\right\|^2 \\ &\leq \|\phi(s,0,x)\|^2 + \left\|\left[G^k(s,0,x)^T - \phi(s,0,x)\right]\right\|^2\end{aligned}$$

As per Lemma 7, $\left\|G^k(s,0,x) - \phi(s,0,x)\right\| \leq c(k)\|\phi(s,0,x)\| \leq Kc(k)\|x\|$. From stability we have $\|\phi(s,0,x)\| \leq K\|x\|$. Hence,

$$V_k(x) = \int_0^\delta \left\|G^k(s,0,x)\right\|^2 ds \leq \delta K^2 \left(1 + c(k)^2\right) \|x\|^2.$$

Therefore the upper boundedness condition is satisfied for any $k \geq 0$ with $\beta = \delta K^2(1 + c(k)^2) > 0$.

Next we consider the strict positivity condition. First we note

$$\|\phi(s,0,x)\|^2 = \left\|G^k(s,0,x) + \left[\phi(s,0,x) - G^k(s,0,x)\right]\right\|^2$$
$$\leq \left\|G^k(s,0,x)\right\|^2 + \left\|\phi(s,0,x) - G^k(s,0,x)\right\|^2$$

which implies

$$\left\|G^k(s,0,x)\right\|^2 \geq \|\phi(s,0,x)\|^2 - \left\|\phi(s,0,x) - G^k(s,0,x)\right\|^2$$

By Lipschitz continuity of f, $\|\phi(s,0,x)\|^2 \geq e^{-2Ls}\|x\|^2$ and $\left\|G^k(s,0,x) - \phi(s,0,x)\right\| \leq Kc(k)\|x\|$. Thus

$$V_k(x) = \int_0^\delta \left\|G^k(s,0,x)\right\|^2 ds \geq \left(\frac{1}{2L}(1-e^{-2L\delta}) - \delta Kc(k)^2\right)\|x\|^2.$$

Therefore for k as defined previously, $\frac{1}{2L}(1-e^{-2L\delta}) - \delta Kc(k)^2 > 0$ and so the positivity condition holds for some $\alpha > 0$.

b) Negativity of the Derivative: Next, we prove the derivative condition. Recall

$$V_k(x) := \int_0^\delta G^k(s,0,x)^T G^k(s,0,x) ds = \int_t^{t+\delta} G^k(s,t,x)^T G^k(s,t,x) ds$$

then since $\nabla V(x(t))^T f(x(t)) = \frac{d}{dt}V(x(t))$, we have by the Leibnitz rule for differentiation of integrals,

$$\frac{d}{dt}V_k(x(t)) = \left[G^k(t+\delta,t,x(t))^T G^k(t+\delta,t,x(t))\right] - \left[G^k(t,t,x(t))^T G^k(t,t,x(t))\right]$$
$$+ \int_t^{t+\delta} 2G^k(s,t,x(t))^T \frac{\partial}{\partial 2} G^k(s,t,x(t)) ds + \int_t^{t+\delta} 2G^k(s,t,x(t))^T \frac{\partial}{\partial 3} G^k(s,t,x(t)) f(x(t)) ds$$
$$= \left\|G^k(\delta,0,x(t))\right\|^2 - \|x(t)\|^2 + \int_t^{t+\delta} 2G^k(s,t,x(t))^T \left[\frac{\partial}{\partial 2} G^k(s,t,x(t)) + \frac{\partial}{\partial 3} G^k(s,t,x(t)) f(x(t))\right] ds$$

where $\frac{\partial}{\partial i} f$ denotes partial differentiation of f with respect to its ith variable. As per Lemma 9, we have

$$\left\|\frac{\partial}{\partial 2} G^k(s,t,x(t)) + \frac{\partial}{\partial 3} G^k(s,t,x(t))^T f(x(t))\right\| \leq \frac{d^k}{T}(K + c(k))\|x(t)\|$$

and as previously noted $\left\|G^k(s,t,x(t))\right\|^2 \le (K^2 e^{-2\lambda(s-t)} + c(k)^2)\,\|x(t)\|^2$. Similarly, $\left\|G^k(s,t,x(t))\right\| \le (K + c(k))\,\|x(t)\|$. We conclude

$$\frac{d}{dt}V_k(x(t)) \le (K^2 e^{-2\lambda\delta} + c(k)^2)\,\|x(t)\|^2 - \|x(t)\|^2 + 2\delta\frac{d^k}{T}(1+c(k))(K+c(k))\,\|x(t)\|^2$$
$$\le \left(K^2 e^{-2\lambda\delta} + c(k)^2 - 1 + 2\delta K\frac{d^k}{T}(1+c(k))(K+c(k))\right)\|x(t)\|^2$$

Therefore, we have strict negativity of the derivative since

$$K^2 e^{-2\lambda\delta} + c(k)^2 + 2\delta\frac{d^k}{T}(1+c(k))(K+c(k))$$
$$= \frac{1}{2} + c(k)^2 + 2\delta K\frac{d^k}{T}(1+c(k))(K+c(k)) < 1$$

Thus $\frac{d}{dt}V_k(x(t)) \le -\gamma\|x(t)\|^2$ for some $\gamma > 0$.

c) Sum of Squares: Since f is polynomial and z is trivially polynomial, $(P^k z)(s,0,x)$ is a polynomial in x and s. Therefore, $V_k(x)$ is a polynomial for any $k > 0$. To show that V is sum-of-squares, we first rewrite the function

$$V(x) = \sum_{i=1}^{N}\int_{iT-T}^{iT}\left[G_i^k(s-iT,0,x)^T G_i^k(s-iT,0,x)\right]ds.$$

Since $G_i^k z$ is a polynomial in all of its arguments, $G_i^k(s-iT,0,x)^T G_i^k(s-iT,0,x)$ is sum-of-squares. It can therefore be represented as $R_i(x)^T Z_i(s)^T Z_i(s) R_i(x)$ for some polynomial vector R_i and matrix of monomial bases Z_i. Then

$$V(x) = \sum_{i=1}^{N} R_i(x)^T \int_{iT-T}^{iT} Z_i(s)^T Z_i(s)\,ds R_i(x) = \sum_{i=1}^{N} R_i(x)^T M_i R_i(x)$$

Where $M_i = \int_{iT-T}^{iT} Z_i(s)^T Z_i(s)ds \ge 0$ is a constant matrix. This proves that V is sum-of-squares since it is a sum of sums-of-squares.

d) Degree Bound: Given a k which satisfies the inequality conditions on $c(k)$, we consider the resulting degree of G^k, and hence, of V^k. If f is a polynomial of degree q, and y is a polynomial of degree d in x, then Py will be a polynomial of degree $\max\{1, dq\}$ in x. Thus since $z = 0$, the degree of $P^k z$ will be q^{k-1}. If $N > 1$, then the degree of G_i^k will be q^{Nk-1}. Thus the maximum degree of the Lyapunov function is $2q^{(Nk-1)}$.

■

VIII. Numerical Illustration

To illustrate the degree bound and hence the complexity of analyzing a nonlinear system, we plot the degree bound versus the exponential convergence rate of the the system. The

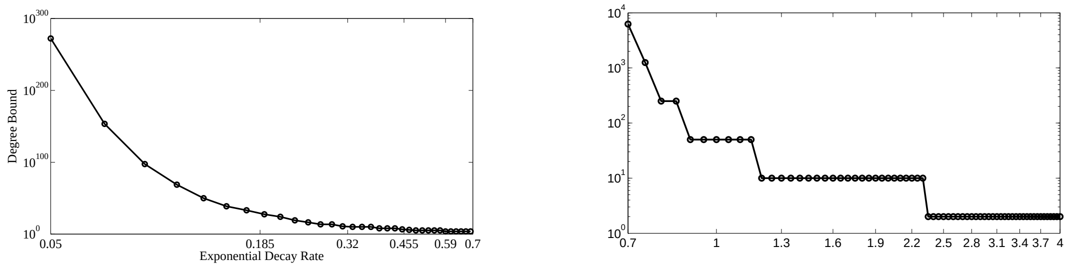

Fig. 2. Degree bound vs. Exponential Convergence Rate for $K = 1.2$, $r = L = 1$, $q = 5$

convergence rate can be viewed as a metric for the accuracy of the sum-of-squares approach: suppose we have a degree bound as a function of convergence rate, $d(\gamma)$. If it is not possible to find a sum-of-squares Lyapunov function of degree $d(\gamma)$ proving stability, then we know that the convergence rate of the system must be less than γ.

As can be seen, as the convergence rate increases, the degree bound decreases super-exponentially, so that at $\gamma = 2.4$, only a quadratic Lyapunov function is required to prove stability. For cases where high accuracy is required, the degree bound increases quickly; scaling approximately as $e^{\frac{1}{\gamma}}$. To reduce the complexity of the problem, in come cases less conservative bounds on the degree can be found by considering the monomial terms in the vector field. If the complexity is still unacceptably high, then one can consider the use of parallel computing: unlike single-core processing, parallel computing power continues to increase exponentially. For a discussion on using parallel computing to solve polynomial optimization problems, we refer to [23].

IX. Conclusion

In this paper, we have used the Picard iteration to construct an approximation to the solution map on arbitrarily long intervals. We have used this approximation to prove that local exponential stability of a polynomial vector field implies the existence of a Lyapunov function which is a sum-of-squares of polynomials. Still unresolved is the fundamental question of whether *globally* stable vector fields will also admit sum-of-squares Lyapunov functions.

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